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A NEW PROOF OF THE AMITSUR-LEVITSKI IDENTITY

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ABSTRACT

We give a simple proof of the Amitsur-Levitzki identity by analysing the powers of matrices with "differential 1-forms" as entries. Using the fact that 2-forms are central the identity is seen to follow from the Cayley-Hamilton theorem.

In 1950 Amitsur and Levitzki [1] proved the following. Let $A_1, A_2, \cdots, A_{2n}$ be $n \times n$ matrices (over a commutative ring). Then

$$(*) \qquad \sum \operatorname{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0,$$

the summation extending over the full symmetric group $\mathrm{Sym}(2n)$.

Their proof is rather long and difficult. It was redone, using graph theory, by Swan [3, 4]. Another proof of (*) was given, in 1958, by Kostant [2] who proved that (*) is equivalent to several non-trivial results in Lie algebras cohomology and group representations.

Here I propose still another proof of (*) which is quite short and elementary. It rests on the following simple facts.

(i) It is enough to prove (*) over a field of characteristic 0. Indeed, the left hand side of (*) is a matrix whose entries are polynomials, with rational integral coefficients, in the entries of the matrices A_i, $i = 1, 2, \cdots, 2n$. If these vanish in any infinite field, they are 0.

(ii) If R is a commutative ring containing the rationals and A an $n \times n$ matrix such that $\mathrm{tr}(A^i) = 0$ for all $i > 0$ then $A^n = 0$. Indeed, this is clear if R is a domain. The general case, which is what we need, is as follows. The Newton formulas on symmetric functions, applied to a generic matrix (that is a matrix with indeterminates as entries) show that the coefficients of the characteristic polynomial are polynomials, with rational coefficients (and with zero constant term) in the traces. Specializing, we obtain what we want.

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If $B_1, B_2, \cdots, B_r$ are $n \times n$ matrices let

$$S_r = S_r(B) = S_r(B_1, \cdots, B_r) = \sum \operatorname{sgn}(\sigma) B_{\sigma(1)} B_{\sigma(2)} \cdots B_{\sigma(r)}$$

where the sum is over $\mathrm{Sym}(r)$.

(iii) If r is even $\operatorname{tr} S_r = 0$. This is part of Lemma 3.4 of Kostant [2] and is really quite trivial: in S_r the trace of each $B_{i_1} B_{i_2} \cdots B_{i_r}$ cancels with the trace of $B_{i_2} \cdots B_{i_r} B_{i_1}$ if r is even.

From now on everything will be over a fixed field K of characteristic 0. Let T be a vector space of dimension $2n$ and let E be the exterior algebra of T. Elements of E are called forms, K is identified as the set of 0-forms and T as 1-forms. The subalgebra of E generated by K and the set of 2-forms is central and we label it R. Identifying $M_n(K) \otimes E$ with $M_n(E)$ we have in a natural way $M_n(K) \subset M_n(R) \subset M_n(E)$.

Let $e_1, e_2, \cdots, e_{2n}$ be an (ordered) basis for T. We write $e_i e_j$ for $e_i \wedge e_j$. Finally let

$$A = A_1 e_1 + A_2 e_2 + \cdots + A_{2n} e_{2n} \in M_n(E).$$

LEMMA. *For every integer $k \geqq 1$*

$$A^k = \sum_{i_1 < i_2 < \cdots < i_k} S_k(A_{i_1}, \cdots, A_{i_k}) e_{i_1} \cdots e_{i_k}.$$

In particular, $A^k = 0$ for $k > 2n$ and

$$A^{2n} = S_{2n}(A) e_1 e_2 \cdots e_{2n}.$$

PROOF. This is immediately verified. ■

Thus (*) is equivalent to $A^{2n} = 0$. But $A^{2n} = (A^2)^n$ and the entries of A^2 are in the commutative ring R. Hence (ii) above applies and the identity (*) would follow if we knew that $\operatorname{Tr}(A^2)^r = 0$ for all $r \geqq 1$. This, however, is shown in (iii).

The proof is now complete.

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